PHYSICAL REVIEW B **81**, 205106 (2010)

Renormalization group for mixed fermion-boson systems

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(Received 26 August 2009; published 7 May 2010)

We formulate a momentum-shell renormalization-group (RG) procedure that can be used in theories containing both bosons and fermions with a Fermi surface. We focus on boson-fermion couplings that are nearly forward scattering, i.e., involving small momentum transfer ($\vec{q} \approx 0$) for the fermions. Special consideration is given to phase space constraints that result from the conservation of momentum and the imposition of ultraviolet cutoffs. For problems where the energy and momentum scale similarly (dynamic exponent $z=1$), we show that more than one formalism can be used and they give equivalent results. When the energy and momentum must scale differently ($z \neq 1$), the procedures available are more limited but a consistent RG scheme can still be formulated. The approach is applicable to a variety of problems, such as itinerant-electron magnets and gauge fields interacting with fermions.

DOI: 10.1103/PhysRevB.81.205106 PACS number(s): 05.10.Cc, 64.60.ae

I. INTRODUCTION

Although the theory of scaling and renormalization has profoundly affected our conceptual understanding of many-body systems, its calculational framework is imperfect and continually evolving. In the end, we are interested in how couplings flow under changes of scale, but a variety of distinct procedures exist, each with its own advantages and drawbacks. An incomplete list of the assortment of programs includes the multiplicative renormalization-group (RG), real-space decimation, functional RG, exact RG, flow equations, and various flavors of ϵ expansion, such as the classic minimal subtraction which expands around $d=4$, or expansions around some other parameter, such as the deviation of the range of the interaction from a suitable reference value. Each method has its own limits of practicality, ease of use, and range of problems to which it may be usefully employed. Wilson's momentum-shell approach[1,2] is an especially popular method in the context of condensed-matter problems. However, in the early 1990s a few people recognized[3–5] that the standard momentum-shell procedure must be modified for problems involving a Fermi surface. A campaign soon followed attempting to understand Fermi-liquid theory from an RG perspective. An excellent and influential summary of the pure-fermion RG can be found in Ref. 6.

Another indication that the RG for fermions required more scrutiny came from the study of quantum criticality in itinerant electron magnets. The usual Hertzian approach[2] uses an auxiliary (Hubbard-Stratonovich) field to decouple the fermion-fermion interaction, thus allowing fermions to be completely integrated out. The resulting effective theory is then expressed in terms of the remaining bosonic auxiliary field, to which standard bosonic RG techniques can be employed. However, because the fermions are gapless, the process of integrating them out may introduce nonanalyticities in the couplings among the remaining bosonic modes.[7,8] It would therefore be important to devise an RG scheme capable of simultaneously handling both gapless bosons and fermions with a Fermi surface.

Besides the critical itinerant magnets, a mixed RG formalism for gapless fermionic-bosonic systems would be quite useful for an assortment of problems. For example, in the context of a gauge field coupled to fermions, several authors[9–12] have developed their own RG schemes for counting dimensions in these mixed theories. All have in common the subdivision of the Fermi surface into a large number of patches but results vary and despite the intervening 15 years since this pioneering work, little progress has been made. The importance of the gauge-fermion problem is historically linked to an interesting path to non-Fermi-liquid behavior.[13,14] More recently, effective gauge theories have appeared in a number of additional contexts in condensed-matter physics.[15]

We should mention in passing a growing body of work on the functional RG which may be adapted for mixed boson-fermion theories.[16–21] This typically requires blending with computational methods and may prove to be a useful framework for understanding realistic material band structures. Our aim here is rather more modest, which is to develop an RG scheme for mixed theories with a high score in the "ease of use" category. This was the chief virtue of the original Wilsonian RG which could quickly identify the relevant and irrelevant operators with minimal effort. One emphasis of this paper will be to carefully consider how to extend Shankar's scheme[6] to include bosons while maintaining the easy-to-use spirit of the Wilsonian approach.

Our primary motivation to consider these issues came from the context of magnetically ordered phases of some itinerant systems. In the case of an antiferromagnetic state of a Kondo lattice, the bosonic magnons, described by a quantum nonlinear sigma model, are coupled to the fermionic quasiparticles near a Fermi surface. In this problem, energy and momentum scale the same way; the dynamic exponent $z=1$. The ferromagnetic counterpart features $z=3$. The RG analysis plays an essential role in understanding the Fermi surfaces in these systems and has been briefly described in our earlier works.[22–24] The purpose of the present work is to explain the details of the method in considerable detail with the hope that the method will be adapted to problems in new physical contexts.

The remainder of the paper is organized as follows. In Sec. II we remind the reader of the essential points of the bosonic Wilson-Hertz scaling. Section III quickly moves on to discuss scaling in fermionic systems, largely paraphrasing what has already been done but emphasizing a slightly different perspective. Section IV describes a method to properly scale in mixed theories when energy and momentum can be given the same scaling dimension, i.e., when $z=1$. This is closest in spirit to the Shankar approach but cannot be generalized to $z\neq 1$. Some of these problems are discussed in Secs. V and VI. In Sec. VI we present an alternative method for arbitrary z that is perhaps less intuitive than that of Sec. IV but has the advantage of being generalizable to $z\neq 1$ while at the same time yielding identical results when $z=1$.

II. BOSON SCALING

The problem we are concerned with can be decomposed into bosonic, fermionic, and interaction terms,

$$\mathcal{S}=\mathcal{S}^f+\mathcal{S}^b+\mathcal{S}_3^{bf}. \tag{1}$$

The bosonic and fermionic pieces can be further divided into quadratic and quartic pieces,

$$\mathcal{S}^b=\mathcal{S}_2^b+\mathcal{S}_4^b, \tag{2}$$

$$\mathcal{S}^f=\mathcal{S}_2^f+\mathcal{S}_4^f. \tag{3}$$

Theories based upon $\mathcal{S}^b$ or $\mathcal{S}^f$ alone have already been subjected to momentum-shell RG analyses; see, for example, Refs. 1, 2, and 6. In this section, we review the Wilson-Hertz scaling procedure for bosons, so we are only concerned with $\mathcal{S}^b$.

In the most general case, the quadratic part of the action can take several different forms depending on the value of z. For example,

$$\mathcal{S}_2^b(z=1)=\int d^dqd\omega\phi^*(q^2+\omega^2)\phi,$$

$$\mathcal{S}_2^b(z=2)=\int d^dqd\omega\phi^*(q^2+\omega)\phi,$$

$$\mathcal{S}_2^b(z=3)=\int d^dqd\omega\phi^*\left(q^2+\frac{\omega}{q}\right)\phi,$$

$$\mathcal{S}_2^b(z=4)=\int d^dqd\omega\phi^*\left(q^2+\frac{\omega}{q^2}\right)\phi. \tag{4}$$

The bosons might represent acoustic phonons, magnons, photons, or some collective mode of an underlying fermionic theory that results after "integrating out" the fermions with an auxiliary field. At this point, we need not be specific. All that matters is that we must design the RG scheme in such a way that $\mathcal{S}_2^b$ remains invariant. The Wilsonian RG for bosons is well known[1] so we only review those elements crucial to the comparisons we wish to make later with the fermionic RG.

Consider a d-dimensional integral in momentum space with a cutoff to high energy and therefore large-q modes; this is denoted by Λ. Let us separate out a thin shell of high-energy modes in the range $\Lambda/s<q<\Lambda$, where $s\gtrapprox 1$,

$$\int^{\Lambda}d^dq\equiv\int d^{d-1}\Omega_{\vec{q}}\int_0^{\Lambda}q^{d-1}dq$$
$$=\int d^{d-1}\Omega_{\vec{q}}\left[\int_0^{\Lambda/s}q^{d-1}dq+\int_{\Lambda/s}^{\Lambda}q^{d-1}dq\right].$$

Here, $q\equiv|\vec{q}|$ is the radial coordinate in (hyper)spherical coordinates and $d^{d-1}\Omega_{\vec{q}}$ represents the measure for integration over all angular variables in $\vec{q}$ space. We have ignored factors of 2π. Mode elimination amounts to simply throwing away the shell integral. To regain the original form of the action only a trivial rescaling of the radial coordinate is needed,

$$q'\equiv sq. \tag{5}$$

This defines the scaling dimension of momentum. In the customary notation, we use square brackets to denote the scaling dimension of any quantity according to

$$A'=s^{[A]}A, \tag{6}$$

where A' is measured in units $s^{[A]}$ times smaller than the units of A. We call $[A]$ the scaling dimension of A. This notation differs from another frequent convention which may claim, for instance, $[\text{volume}]=L^3$, where L is some length scale. We prefer our notation since it means a coupling g is relevant when $[g]>0$, irrelevant when $[g]<0$, and marginal when $[g]=0$.

In this notation, an equivalent statement to Eq. (5) is simply

$$[q]=1. \tag{7}$$

Using this form of momentum scaling in the integral leads to

$$\int d^{d-1}\Omega_{\vec{q}}\int_0^{\Lambda}s^{-(d-1)}q'^{d-1}s^{-1}dq'=s^{-d}\int^{\Lambda}(d^dq)'.$$

We conclude that the scaling dimension of the measure is given by

$$[d^dq]=d[q]=d. \tag{8}$$

Note that rescaling the radial variable, q, is the same as rescaling all the components of $\vec{q}$ since $q=\sqrt{\Sigma_\alpha^d q_\alpha^2}$. For this to be consistent with $q'=sq$, we must have $q_\alpha=sq_\alpha$ for all components $\alpha\in\{x,y,z,\ldots\}$. This is an important difference from the fermionic case to be discussed later and results from the simple fact that the coordinate origin here is a single point rather than an extended surface.

Let us apply this mode elimination and rescaling to the quadratic part of the boson action, taking the case $z=3$ as an example,

$$\mathcal{S}_2^b(z=3) = \int^{\Lambda/s} d^dq d\omega |\phi|^2\left(q^2 + \frac{\omega}{q}\right) + (\text{high energy})$$
$$\approx s^{-d[q]-[\omega]}\int^{\Lambda} d^dq' d\omega' |\phi(s^{-[q]}\vec{q}', s^{-[\omega]}i\omega')|^2$$
$$\times\left(s^{-2[q]}q'^2 + s^{-[\omega]+[q]}\frac{\omega'}{q'}\right), \quad (9)$$

where we discarded high-energy modes and used Eq. (5) and $\omega' \equiv s^{[\omega]}\omega$. We wish to scale the terms in the parentheses identically, so we must have $-2[q]=-[\omega]+[q]$ thus fixing the relationship between the scaling dimensions of energy and momentum,

$$[q] = [\omega]/z, \quad (10)$$

where our example considered $z=3$ explicitly.

The final step is wave-function renormalization which can be implemented by defining a new field ϕ' according to

$$\phi'(q', i\omega') \equiv s^{-\{(d+2)[q]+[\omega]\}/2}\phi(s^{-[q]}q', s^{-[\omega]}i\omega'). \quad (11)$$

Equivalently, we might say that the boson field has a scaling dimension given by

$$[\phi] = -\frac{[q]}{2}(d+z+2), \quad (12)$$

where we used $[\omega]=z[q]$. Although customary Eq. (12) is slightly misleading. The replacement of ϕ by $s^{-[\phi]}\phi'$ in analyzing interaction terms should only be done when the arguments of the field transform according to Eq. (11). Equation (61) provides an example where the arguments of the field are transformed in a very different fashion. From Eq. (11) we see that the boson field appears to take the form of a generalized homogeneous function. We do not delve into this issue further but merely note that Eq. (11) is a very specific type of substitution that needs to be implemented in this strict form. Scale invariance of $\mathcal{S}_2^b$ has imposed a transformation property on the field, specified in Eq. (11), under the particular coordinate transformation $q'=s^{[q]}q$ and $\omega'=s^{[\omega]}\omega$ with $[q]=[\omega]/z$.

Now that we know how to scale momentum from Eq. (7), energy from Eq. (10), and the field from Eq. (11), we are ready to analyze the four-boson interaction term,

$$\mathcal{S}_4^b = u_b\int d^dq_4 d\omega_4 d^dq_3 d\omega_3 d^dq_2 d\omega_2 d^dq_1 d\omega_1$$
$$\times\phi(\vec{q}_4, i\omega_4)\phi(\vec{q}_3, i\omega_3)\phi(\vec{q}_2, i\omega_2)\phi(\vec{q}_1, i\omega_1)$$
$$\times\Theta(\Lambda - |\vec{q}_4|)\Theta(\Lambda - |\vec{q}_3|)\Theta(\Lambda - |\vec{q}_2|)\Theta(\Lambda - |\vec{q}_1|)$$
$$\times\delta^{(d)}(\vec{q}_4 + \vec{q}_3 - \vec{q}_2 - \vec{q}_1)\delta(\omega_4 + \omega_3 - \omega_2 - \omega_1). \quad (13)$$

The δ functions enforce the conservation of energy and momentum while the Θ functions define the cutoffs for the effective-field theory (in principle, energy cutoffs should also be written but this is understood).

To determine the scaling dimension of u_b at the tree level, we first separate the integrations into low- and high-energy modes [i.e., $\Theta(\Lambda-|\vec{q}_i|)=\Theta(\Lambda/s-|\vec{q}_i|)+\Theta(|\vec{q}_i|-\Lambda/s)\Theta(\Lambda-|\vec{q}_i|)$], then discard the high-energy shell. There is some freedom in choosing the shape of the shell which can take some curious forms for the purpose of simplifying calculations. See the discussion by Hertz.[2]

After rescaling according to Eqs. (7), (10), and (12), we find

$$\mathcal{S}_4^b = s^{4-d-z}u_b\int d^dq_3' d\omega_3' d^dq_2' d\omega_2' d^dq_1' d\omega_1'$$
$$\times\phi(\vec{q}_4', i\omega_4')\phi(\vec{q}_3', i\omega_3')\phi(\vec{q}_2', i\omega_2')\phi(\vec{q}_1', i\omega_1')$$
$$\times\Theta(\Lambda - |\vec{q}_4'|)\Theta(\Lambda - |\vec{q}_3'|)\Theta(\Lambda - |\vec{q}_2'|)\Theta(\Lambda - |\vec{q}_1'|)$$
$$\times\delta^{(d)}(\vec{q}_4' + \vec{q}_3' - \vec{q}_2' - \vec{q}_1')\delta(\omega_4' + \omega_3' - \omega_2' - \omega_1'), \quad (14)$$

which tells us that $u_b' \equiv s^{4-d-z}u_b$, or equivalently

$$[u_b] = 4 - (d+z). \quad (15)$$

This yields a quick way to determine when the four-boson interaction term $u_b\int\phi^4$ is relevant or irrelevant based on the dimensionality of the problem and the value of z. Historically, this result provided some early intuition about quantum phase transitions which can behave like classical phase transitions but in a different number of effective dimensions: $d_{\text{eff}}=d+z$. Although the theory was originally devised to address questions about itinerant quantum critical magnets,[2,25,26] some problems have been encountered with this approach.[7,8] Part of the problem could be that the theory is completely bosonic, despite the underlying fermionic nature of the system. It is therefore desirable to develop an RG formalism that includes fermions with a Fermi surface.

III. FERMION SCALING: SHANKAR'S RG

For fermions, the quadratic part of the action is given by

$$\mathcal{S}_2^f = \int d^dK d\epsilon\bar{\psi}(i\epsilon - \xi_{\vec{K}})\psi. \quad (16)$$

To define a scaling scheme that leaves $\mathcal{S}_2^f$ scale invariant, we now review the formulation of the fermionic RG.[6] We shall use Shankar's notation and label momenta measured with respect to the Brillouin-zone center with a capital letter $\vec{K}=(K_x, K_y, \ldots)$. In contrast to the bosonic case, low-energy modes live near an extended surface (the Fermi surface) rather than a single point (the Brillouin-zone center). For a spherical Fermi surface, a high-energy cutoff can be implemented on $\vec{K}$ integrals as follows:

$$\int^{\Lambda} d^dK \equiv \int d^{d-1}\Omega_{\vec{K}}\int_{K_F-\Lambda}^{K_F+\Lambda} K^{d-1}dK,$$

where Λ is an ultraviolet cutoff but we still insist $\Lambda \ll K_F$. Here, $d^{d-1}\Omega_{\vec{K}}$ represents the measure for integration over all angular coordinates in $\vec{K}$ space while $K\equiv|\vec{K}|$ is the radial coordinate. Usually, we work at fixed fermion density which, by Luttinger's theorem, dictates that we design our scaling scheme in such a way that the Fermi volume remains invariant. To preserve the Fermi surface under rescaling we cannot simply scale the radial coordinate as we did in the bosonic case. To see this, observe that after mode elimination the expression we wish to rescale is given by

$$\int^{\Lambda/s} d^d K \equiv \int d^{d-1}\Omega_{\vec{K}} \int_{K_F-\Lambda/s}^{K_F+\Lambda/s} K^{d-1} dK. \quad (17)$$

Clearly, no simple rescaling of K will return the integral to its original form. This is the principle disparity between the fermionic and bosonic RG. To make progress we define the lower case letter $k \equiv |\vec{K}| - K_F$. Note that $k=0$ corresponds to $\xi_{\vec{K}} = \epsilon_{\vec{K}} - \mu = 0$ since $\xi_{\vec{K}} = \frac{K^2-K_F^2}{2m} \approx v_F(K-K_F) = v_F k$. Small k corresponds to low energy whereas small K does not. Such a change in variables greatly facilitates rescaling,

$$\int d^{d-1}\Omega_{\vec{K}} \int_{-\Lambda/s}^{\Lambda/s} (K_F+k)^{d-1} dk$$
$$= K_F^{d-1} \int d^{d-1}\Omega_{\vec{K}} \int_{-\Lambda/s}^{\Lambda/s} \left(1 + \frac{k}{K_F}\right)^{d-1} dk$$
$$\approx K_F^{d-1} \int d^{d-1}\Omega_{\vec{K}} \int_{-\Lambda/s}^{\Lambda/s} dk. \quad (18)$$

We have neglected certain terms above for two reasons: they are of order Λ/K_F relative to what has been kept and they are less relevant in the RG sense. To see the latter, note that the integral can be restored to its original form with the simple rescaling $k' = sk$. This determines the scaling dimension

$$[k] = 1. \quad (19)$$

Note that the variable k is not a vector, nor is it a radial coordinate since it can take negative values. Later, we will discuss another scheme, which we call "patching," that decomposes the momenta into components parallel ($\vec{k}_\parallel$) and perpendicular ($k_\perp$) to the Fermi-surface normal. To make later contrast with the patching scheme of Sec. VI, which uses local coordinates for each patch, we will call the present approach the "global coordinate" scheme.

To further emphasize the dissimilarity between the fermionic and bosonic cases, observe that after the rescaling of Eq. (19),

$$\int^{\Lambda/s} d^d K \approx K_F^{d-1} \int d^{d-1}\Omega_{\vec{K}} \int_{-\Lambda}^{\Lambda} s^{-1} dk' = s^{-1} \int^{\Lambda} (d^d K)', \quad (20)$$

which implies that, effectively,

$$[d^d K] = 1. \quad (21)$$

This stands in sharp contrast to the bosonic case in Eq. (8). Here, the angular variables are truly untouched after rescaling which is necessary to maintain the Fermi surface. Unfortunately, the straightforward transformation $k' = sk$ does not translate into a simple transformation on the components of $\vec{K}$. Care must therefore be exercised to write all expressions in terms of k before the scaling procedure can begin. For example, after mode elimination and rescaling of energy and momentum, the quadratic part of the fermionic action is given by

$$\mathcal{S}_2^f \propto s^{-3} \int dk' d\epsilon' \bar{\psi}(K_F + s^{-1}k', s^{-1}i\epsilon')$$
$$\times [s^{-[\epsilon]} i\epsilon' - v_F s^{-[k]} k'] \psi(K_F + s^{-1}k', s^{-1}i\epsilon').$$

If we wish to scale both of the terms inside the square brackets identically, we must choose

$$[k] = [\epsilon], \quad (22)$$

thus fixing the relationship between the scaling dimensions of energy and momentum. For convenience we can set this value equal to 1, as in Eq. (19). Compare this to Eq. (10).

In order to make $\mathcal{S}_2^f$ invariant to the RG transformation we must demand that the fermion field obeys

$$s^{-3/2}\psi(K_F + s^{-1}k', s^{-1}i\epsilon') = \psi'(K_F + k', i\epsilon'), \quad (23)$$

where we have not explicitly written the dependence of ψ on angular variables since these do not scale. Equation (23) tells us two important things. First, the dimension of the fermion field is simply

$$[\psi] = -3/2. \quad (24)$$

Second, the RG transformation of the fermion field does *not* take the form of a generalized homogeneous function as was the case for the bosonic field; see Eq. (11). The momentum argument of the fermion field $\vec{K}$ has a magnitude equal to the Fermi wave vector plus a small deviation: $K = K_F + k$. Only the deviation k scales while K_F remains constant. This important difference from the bosonic case will be discussed further in Sec. V.

The story so far seems relatively elementary but the true subtleties materialize when we try to determine the dimension of the ψ^4 coupling function u_f based on the dimension assignments required to make $\mathcal{S}_2^f$ scale invariant. The quartic part of the action can be written as[6]

$$\mathcal{S}_4^f = \prod_{i=1}^{4} \int^{\Lambda} d^d K_i \int d\epsilon_i \bar{\delta}^{(d)}(\vec{K}_1 + \vec{K}_2 - \vec{K}_3 - \vec{K}_4)$$
$$\times \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4)$$
$$\times \bar{\psi}(4)\bar{\psi}(3)\psi(2)\psi(1) u_f(4,3,2,1). \quad (25)$$

The δ functions explicitly enforce the conservation of energy and momentum (up to a reciprocal-lattice vector). We might integrate one of the energies and momenta, say $(\vec{K}_4, \epsilon_4)$, against the delta function to yield an integral over three independent sets $(\vec{K}_1, \epsilon_1)$, $(\vec{K}_2, \epsilon_2)$, and $(\vec{K}_3, \epsilon_3)$,

$$\mathcal{S}_4^f = \prod_{i=1}^{3} \int^{\Lambda} d^d K_i \int d\epsilon_i \bar{\psi}(1+2-3)\bar{\psi}(3)\psi(2)\psi(1)$$
$$\times u_f(1+2-3,3,2,1) \quad \text{(wrong)}. \quad (26)$$

But this expression is not quite right. The problem is that not all momentum-conserving processes should be included in the low-energy effective-field theory. We must respect the cutoff imposed on the quadratic part of the action, which only allows excursion into states within a distance $\pm\Lambda$ of the Fermi surface. Imposing a cutoff amounts to constraining the momentum integrals. Until now, we have implemented the cutoff constraints by writing them explicitly in the limits of integration, but let us re-express them as

$$\int^{\Lambda} d^d K_i = \int d^d K_i \Theta(\Lambda - |k_i|), \tag{27}$$

where, as usual, $k_i \equiv |\vec{K}_i| - K_F$. With all momentum integrals written in this way, we can safely use the δ functions to eliminate one variable, say $\vec{K}_4$ and ϵ_4,

$$\begin{aligned} \mathcal{S}_4^f &= \prod_{i=1}^{3} \int d^d K_i \int d\epsilon_i \bar{\psi}(1+2-3)\bar{\psi}(3)\psi(2)\psi(1) \\ &\quad \times u_f(1+2-3,3,2,1)\Theta(\Lambda - |k_1|)\Theta(\Lambda \\ &\quad - |k_2|)\Theta(\Lambda - |k_3|)\Theta(\Lambda - |\mathcal{K}_4|) \\ &= \prod_{i=1}^{3} \int^{\Lambda} d^d K_i \int d\epsilon_i \bar{\psi}(1+2-3)\bar{\psi}(3)\psi(2)\psi(1) \\ &\quad \times u_f(1+2-3,3,2,1)\Theta(\Lambda - |\mathcal{K}_4|). \end{aligned} \tag{28}$$

The constraints on $\vec{K}_1$, $\vec{K}_2$, and $\vec{K}_3$ have been put back in the limits of integration, but we have the additional constraint $|\mathcal{K}_4| < \Lambda$, where

$$\mathcal{K}_4 \equiv |\vec{K}_3 - \vec{K}_2 - \vec{K}_1| - K_F. \tag{29}$$

Once we have conserved momentum, $\vec{K}_4$ is no longer an independent variable, so we use the notation $\mathcal{K}_4$ to represent the combination of variables specified in Eq. (29).

We can implement the constraint embodied in $\Theta(\Lambda - |\mathcal{K}_4|)$ in a number of ways. One way is to allow $\vec{K}_1$ and $\vec{K}_2$ to range anywhere inside the annuli defined by $-\Lambda < k_1, k_2 < \Lambda$ but restrict $\vec{K}_3$ as appropriate to satisfy $|\mathcal{K}_4| < \Lambda$. The outcome of a proper phase space analysis shows that once $\vec{K}_1$ and $\vec{K}_2$ have been chosen, the angle for $\vec{K}_3$ is highly constrained.[6]

To see this in more detail, observe that to leading order in Λ / K_F,

$$\mathcal{K}_4 \approx K_F(|\vec{\Delta}| - 1), \tag{30}$$

where $\vec{\Delta} \equiv \hat{K}_1 + \hat{K}_2 - \hat{K}_3$, and where the $\hat{K}_i$ are unit vectors, each pointing in the direction of $\vec{K}_i$. Note that $\vec{\Delta}$ is not itself a unit vector since

$$|\vec{\Delta}| = \sqrt{2}\left[\frac{3}{2} + \hat{K}_1 \cdot \hat{K}_2 - \hat{K}_1 \cdot \hat{K}_3 - \hat{K}_2 \cdot \hat{K}_3\right]^{1/2}, \tag{31}$$

a result we will use in Sec. V. After mode elimination, the momentum integrals become

$$\begin{aligned} &\prod_{i=1}^{3} \int d^d K_i \Theta(\Lambda/s - K_F||\vec{\Delta}| - 1|) \\ &\quad = \prod_{i=1}^{3} \int d^d K_i \Theta(\Lambda - sK_F||\vec{\Delta}| - 1|). \end{aligned} \tag{32}$$

Simply rescaling $k_i' = sk_i$ is not sufficient to regain the original form of the action for generic values of k_i. The obvious snag is the annoying way the Θ function transforms. For general values of the momenta k_i, the Θ function is clearly not invariant to the renormalization-group transformation. Consequently, we are not technically entitled to compare the coupling before and after, so we do not know the RG flow. The way out of this dilemma is first to understand the circumstances under which the Θ function is invariant, and then to see what might be happening for more generic cases by considering a soft cutoff.

First, note that when $|\vec{\Delta}| = 1$ (i.e., $\mathcal{K}_4 = 0$) the Θ function is always form invariant since $\Theta(\Lambda) = \Theta(\Lambda/s)$. The condition $|\vec{\Delta}| = 1$ can be fulfilled in three different ways,

$$\text{(i)}\ \vec{K}_1 = \vec{K}_3 \quad \text{and} \quad \vec{K}_2 = \vec{K}_4, \tag{33}$$

$$\text{(ii)}\ \vec{K}_2 = \vec{K}_3 \quad \text{and} \quad \vec{K}_1 = \vec{K}_4, \tag{34}$$

$$\text{(iii)}\ \vec{K}_1 = -\vec{K}_2 \quad \text{and} \quad \vec{K}_3 = -\vec{K}_4. \tag{35}$$

For these values of the momenta, the rescaling $k_i' = sk_i$ works flawlessly because the Θ function is form invariant under these restrictions. We are now allowed to compare the coupling before and after. Since $[dk_1 dk_2 dk_3 d\epsilon_1 d\epsilon_2 d\epsilon_3] = 6$ and $[\psi^4] = -6$, we conclude that, at the tree level, the most relevant pieces of u_f are marginal. This important result is at the heart of Fermi-liquid theory but is expressed by the simple equation,

$$[u_f] = 0. \tag{36}$$

In Shankar's notation, cases (i) and (ii) correspond to $u_f = \pm F$ and case (iii) $u_f = V$. It has also been shown[6] that case (i) remains marginal beyond the tree level, while loop corrections in case (iii) lead to a marginally relevant coupling for certain angular momentum channels, indicative of the BCS instability.

Let us understand in more detail the circumstances under which the Θ function is always form invariant. In particular, we want to stress that the condition $\mathcal{K}_4 = 0$ is conceptually different from the limit $\Lambda / K_F \to 0$. To see this, let us rewrite the equation $\mathcal{K}_4 = 0$ as follows:

$$|\vec{K}_3 - \vec{K}_2 - \vec{K}_1| = K_F. \tag{37}$$

Note that Eq. (29) is slightly more accurate than Eq. (30). Next, define $\vec{P} \equiv \vec{K}_1 + \vec{K}_2$ which obviously gives

$$|\vec{K}_3 - \vec{P}| = K_F. \tag{38}$$

This says that the vector joining the tip of $\vec{K}_3$ to the tip of $\vec{P}$ must have magnitude precisely equal to K_F. Figure 1 depicts the situation. Geometrically, the choices available to $\vec{K}_3$ once $\vec{K}_2$ and $\vec{K}_1$ have been selected are given by the thick gray lines in the figure. Notice that while k_3 can still take any values $-\Lambda < k_3 < \Lambda$ within the annulus, the angle of $\vec{K}_3$ has become highly constrained. However, it is clear that even when $\mathcal{K}_4 = 0$ the value of Λ / K_F can still be nonzero.

We now know the dimension $[u_f]$ when we restrict to $\mathcal{K}_4 = 0$. However, the three cases corresponding to $\mathcal{K}_4 = 0$ constitute only a small portion of $(\vec{K}_1, \vec{K}_2, \vec{K}_3)$ space. To see what

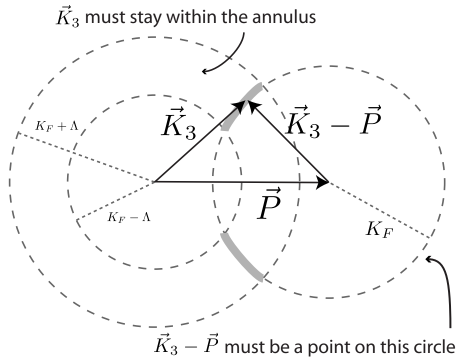

FIG. 1. Once $\vec{K}_1$ and $\vec{K}_2$ have been chosen the conservation of momentum and the requirement that all $\vec{K}_i$ respect the cutoff of the field theory strongly constrains the phase space available to $\vec{K}_3$. Shown here is the limit $\mathcal{K}_4=0$. Even in this limit, there is some flexibility in the choices available to $\vec{K}_3$, as depicted by the thick gray lines. While the magnitude of k_3 can still fall anywhere in the range $-\Lambda<k_3<\Lambda$, the angles available to $\vec{K}_3$ are highly limited. If we were to further take the limit $\Lambda/K_F\to 0$, the gray lines would shrink to points. Note that $\vec{P}$ is defined as the sum of $\vec{K}_1$ and $\vec{K}_2$ but the latter are not drawn to avoid clutter.

happens to the coupling function $u(3,2,1)$ for other values of momenta, Shankar had the insight to employ a soft cutoff: $\Theta(\Lambda-|k_i|)\approx e^{-|k_i|/\Lambda}$. Using this device, the rescaled cutoff for arbitrary k_i values becomes

$$\Theta(\Lambda - sK_F||\vec{\Delta}|-1|) \approx e^{-sN_\Lambda||\vec{\Delta}|-1|} = e^{-N_\Lambda||\vec{\Delta}|-1|}e^{-(s-1)N_\Lambda||\vec{\Delta}|-1|},$$

where we have defined the large parameter $N_\Lambda \equiv K_F/\Lambda$ (generally, we have the hierarchy $k<\Lambda \ll K_F$, which means $N_\Lambda \gg 1$). We choose to write the cutoff in this way because then clearly when $|\vec{\Delta}|=1$, corresponding to the three cases listed above, the cutoff becomes a simple factor of unity. For any $|\vec{\Delta}|\neq 1$, which means all other values of the k_i, the cutoff $\to 0$ in the limit $N_\Lambda\to\infty$ provided $s>1$. While we do not know how the coupling function $u(3,2,1)$ scales for values of the momenta where $|\vec{\Delta}|\neq 1$, it does not matter because these couplings will be exponentially suppressed in the limit $1/N_\Lambda=\Lambda/K_F\to 0$.

Note that the condition $|\vec{\Delta}|=1$ is simply the statement that $\mathcal{K}_4$ should not scale. Indeed, it means $\mathcal{K}_4=0$. Only when $|\vec{\Delta}|=1$ is the relation $\mathcal{K}_4'=s\mathcal{K}_4$ satisfied, albeit trivially. In fact, that is how we identified the condition $|\vec{\Delta}|=1$, being the only combination of $\vec{K}_3$, $\vec{K}_2$, and $\vec{K}_1$, where $\Theta(\Lambda-|\mathcal{K}_4|)$ can be rescaled to take its original form after mode elimination. This useful interpretation will be used again later when we extend the formalism to include bosons.

Before moving on, we need to make another observation about the pure-fermion RG that will be important to later generalizations. We have shown how to find the dimension of the coupling function $u_f(3,2,1)$ for those values of momentum that satisfy $\mathcal{K}_4=0$ (i.e., $|\hat{\Delta}|=1$) corresponding to forward, exchange, and Cooper scattering. To be pedantic, this phase space restriction should be incorporated into the form of the coupling,

$$\mathcal{S}_4^f = \prod_{i=1}^{3}\int^\Lambda d^dK_i \int d\epsilon_i \bar{\psi}(1+2-3)\bar{\psi}(3)\psi(2)\psi(1) \times u_f(1+2-3,3,2,1)\Theta(\Lambda-|\mathcal{K}_4|)\delta(\mathcal{K}_4). \quad (39)$$

Note that $\Theta(\Lambda)=1$ always since $\Lambda>0$. As seen in Fig. 1, the insertion of $\delta(\mathcal{K}_4)$ does not affect the freedom of $\vec{K}_1$ or $\vec{K}_2$ at all, nor does it affect the magnitude of $\vec{K}_3$ so long as $-\Lambda<|\vec{K}_3|-K_F<\Lambda$. However, the angle of $\vec{K}_3$ is highly restricted to the two gray regions of the figure as a resulting of inserting $\delta(\mathcal{K}_4)$. We may therefore implement the constraint (in $d=2$) by

$$\delta(\mathcal{K}_4) \to \delta(|\vec{\Delta}|-1)/K_F = [\delta(\theta_3-\theta_1)+\delta(\theta_3-\theta_2)]/K_F. \quad (40)$$

A similar expression can be written in $d=3$. Since angles do not scale in this scheme, whether or not we insert this factor into $\mathcal{S}_4^f$ will have *no effect* on the value of the dimension of u_f. Shankar's result of marginality, $[u_f]=0$, still holds. We mention this issue because generalizing the method to include bosons will not result in so happy a circumstance. We turn to this case next.

IV. BOSON+FERMION SCALING

We are finally ready to incorporate bosons. Consider the following interaction term involving two fermions and one boson:

$$\mathcal{S}_3^{bf} = \int d^dK_1 d^dK_2 d^dq\, g(\vec{K}_1,\vec{K}_2,\vec{q})\bar{\psi}_{\vec{K}_2}\psi_{\vec{K}_1}\phi_{\vec{q}} \times \delta^{(d)}(\vec{K}_2-\vec{K}_1-\vec{q})\Theta(\Lambda-|k_1|)\Theta(\Lambda-|k_2|) \times\Theta(\Lambda-|\vec{q}|), \quad (41)$$

g is the coupling function which plays the same role as u_f in the four-fermion problem. (For definiteness, we focus our discussion on Yukawa type of fermion-boson coupling, but our method can be readily extended to the general cases of "x fermion-y boson" couplings.) For simplicity we have suppressed frequency integrals and assumed $\Lambda_b\sim\Lambda_f\sim\Lambda$. To conserve momentum we have two choices: use the δ function to eliminate a fermionic momentum $\vec{K}_i$ or the bosonic momentum $\vec{q}$. This gives either

$$\int^\Lambda d^dK d^dq[\bar{\psi}_{\vec{K}+\vec{q}}\psi_{\vec{K}}\phi_{\vec{q}}g(\vec{K},\vec{q})\Theta(\Lambda-|\mathcal{K}_2|)] \quad (42)$$

or

$$\int^\Lambda d^dK_1 d^dK_2[\bar{\psi}_{\vec{K}_2}\psi_{\vec{K}_1}\phi_{K_2-K_1}g(\vec{K}_2,\vec{K}_1)\Theta(\Lambda-|\mathcal{Q}|)], \quad (43)$$

where some of the cutoff constraints have been put back in the limits of integrations, and where we have defined

$$\mathcal{K}_2 \equiv |\vec{K}+\vec{q}|-K_F, \quad (44)$$

$$\vec{\mathcal{Q}} \equiv \vec{K}_1 - \vec{K}_2. \tag{45}$$

This is analogous to what we did for the pure-fermion problem; see Eqs. (28) and (29). Note that because we integrated against the delta functions, momentum and energy are already explicitly conserved. In Eq. (42), $\vec{K}_2$ is no longer an independent variable, so we use the symbol $\mathcal{K}_2$ to represent the combination of variables specified in Eq. (44). Likewise, $\vec{q}$ is not an independent variable in Eq. (43), so we use $\vec{\mathcal{Q}}$ as shorthand for the momentum transfer, as specified in Eq. (45). This mirrors the development of the pure-fermion case.

Unlike the pure-fermion problem, we now appear to have two different choices for expressing the boson-fermion coupling. Equation (42) involves the boson-fermion coupling function $g(\vec{K},\vec{q})$ while Eq. (43) contains $g(\vec{K}_2,\vec{K}_1)$. We defer a discussion of the resolution of this choice to Sec. V. Here, we simply point out that a consistent scheme can only be found for Eq. (42) and we adopt this choice for the remainder of this section.

Although momentum is conserved, just like the pure-fermion case, not all momentum conserving processes are allowed because some might fall outside the high-energy cutoffs. We must further restrict the coupling function g with the constraint $\Theta(\Lambda-|\mathcal{K}_2|)$. Unfortunately, this quantity only scales in a simple way when $z=1$. Let us briefly explain the problem.

Recall from the form of $\mathcal{S}_2^f$ that we have the relation $[k]=[\epsilon]$ while $\mathcal{S}_2^b$ demands $[q]=[\omega]/z$ for general values of z [see Eqs. (10) and (22)]. In addition, since we want to scale fermions and bosons at the same time, we choose to scale the energies the same way, that is: $[\omega]=[\epsilon]$. For convenience, we set the scaling dimension of energy to unity: $[\omega]=[\epsilon]=1$. Any other value would change all scaling dimensions by the same multiplicative factor but their relative dimensions would be unaffected. Using this prescription we find

$$[\epsilon]=[k]=[\omega]=1,$$

$$[q]=[\omega]/z=\frac{1}{z},$$

$$[\psi]=-\frac{3}{2},$$

$$[\phi]=-\frac{d+z+2}{2z}. \tag{46}$$

Mode elimination and rescaling according to this scheme leads to the following interaction term (we reinstate the energy integrals):

$$\mathcal{S}_3^{bf}=s^{z+2-d/2z}g\int^{\Lambda} d^dq'\,dk'\,d^{d-1}\Omega_{\vec{K}}d\epsilon'\,d\omega'\,\bar{\psi}'\psi'\phi' \times\Theta(\Lambda/s-|\mathcal{K}_2|). \tag{47}$$

The reason why we have Λ/s rather than $\Lambda/s^{1/z}$ is because this constraint comes from the restriction on the momentum integration of $k_2\equiv|\vec{K}_2|-K_F$ in Eq. (41), which scales like a fermion.

Let us rewrite the expression involved in the Θ function,

$$\begin{aligned}|\mathcal{K}_2| &= |\vec{K}+\vec{q}|-K_F=[(K_F+k)^2+q^2 \\ &\quad +2(K_F+k)q\cos\theta_{Kq}]^{1/2}-K_F \\ &\approx K_F\left[1+\frac{2}{K_F}(k+q\cos\theta_{Kq})\right]^{1/2}-K_F \\ &\approx k+q\cos\theta_{Kq},\end{aligned} \tag{48}$$

which is valid to leading order in Λ/K_F, and where

$$\cos\theta_{Kq}=\hat{K}\cdot\hat{q} \tag{49}$$

$$\equiv\begin{cases}\cos(\theta_K-\theta_q)\\ \cos\theta_K\cos\theta_q+\sin\theta_K\sin\theta_q\cos(\varphi_K-\varphi_q)\end{cases} \tag{50}$$

in $d=2$ and $d=3$, respectively. Equation (47) now becomes

$$\begin{aligned}\mathcal{S}_3^{bf} &= s^{z+2-d/2z}g\int^{\Lambda} d^dq'\,dk'\,d^{d-1}\Omega_{\vec{K}}d\epsilon'\,d\omega'\,\bar{\psi}'\psi'\phi' \\ &\quad\times\Theta(\Lambda/s-|k+q\cos\theta_{Kq}|) \\ &= s^{z+2-d/2z}g\int^{\Lambda} d^dq'\,dk'\,d^{d-1}\Omega_{\vec{K}}d\epsilon'\,d\omega'\,\bar{\psi}'\psi'\phi' \\ &\quad\times\Theta(\Lambda-|k'+s^{(z-1)/z}q'\cos\theta_{Kq}|),\end{aligned} \tag{51}$$

where $k'=sk$, $q'=s^{1/z}q$, $\epsilon'=s\epsilon$, and $\omega'=s\omega$. Clearly, for generic values of z the Θ function does not return to its original form after the renormalization-group transformation. We should be pleased, however, that in the special case $z=1$, the Θ function is form invariant,

$$\begin{aligned}\Theta(\Lambda/s-|\vec{\mathcal{K}}_2|) &\approx \Theta(\Lambda/s-|k+q\cos\theta_{Kq}|) \\ &= \Theta(\Lambda-s|k+q\cos\theta_{Kq}|) \\ &= \Theta(\Lambda-|k'+q'\cos\theta_{Kq}|).\end{aligned} \tag{52}$$

The boson-fermion coupling can now be written as

$$\mathcal{S}_3^{bf}=s^{(3-d)/2}g\int^{\Lambda} d^dq'\,dk'\,d^{d-1}\Omega_{\vec{K}}d\epsilon'\,d\omega'\,\bar{\psi}'\psi'\phi' \times\Theta(\Lambda-|k'+q'\cos\theta_{Kq}|) \tag{53}$$

and we can identify

$$g'\equiv s^{(3-d)/2}g, \tag{54}$$

which is equivalent to

$$[g]=(3-d)/2. \tag{55}$$

This is one of the central results of this paper. The coupling is marginal in $d=3$ and relevant in $d=2$. Of course, this result depends on the choice of field dimensions; Eq. (46) with $z=1$. In application to an antiferromagnetic Kondo lattice, we have previously developed a model where the boson dimension is $-d$ [rather than Eq. (46)] and used the scheme explained here to show that the boson-fermion coupling is exactly marginal in that case.[22]

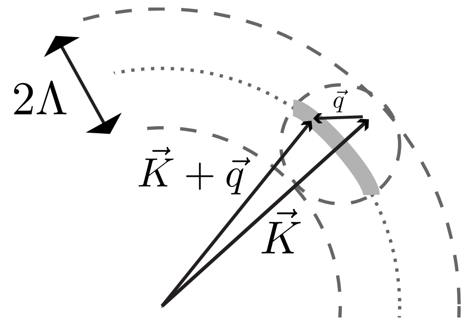

FIG. 2. $\vec{K}$ must stay within the annulus while $\vec{q}$ must stay inside the little circle of radius Λ. Under the restriction $\mathcal{K}_2=0$, the sum $\vec{K}+\vec{q}$ must sit precisely on the Fermi surface. The only phase space that satisfies $\mathcal{K}_2=0$ is the thick gray line which represents a small patch on the Fermi surface of size $O(\Lambda^{d-1})$. Clearly, the limit $\mathcal{K}_2=0$ is not the same as $\Lambda/K_F=0$ since the latter would shrink the gray patch to a point.

Equation (55) is only valid when $z=1$ because only then is the Θ-function form invariant. What can be done when $z\neq 1$? This question is particularly pertinent to the controversy surrounding the renormalization of a gauge field coupled to a fermion with a Fermi surface.[9–12] It is also germane to the ferromagnetic phase of heavy-fermion systems.[24] Section VI will present a different scheme that is applicable to problems with arbitrary values of z and also reproduces Eq. (55) when $z=1$. Here, we merely explain why the present scheme fails when $z\neq 1$.

We have actually already seen the problem in Eq. (51), where it is obvious that the Θ function is not form invariant. This is similar to the dilemma we encountered in the pure-fermion problem, as seen in Eq. (32). To make progress, we try the same strategy used in the pure-fermion problem where we restricted our consideration to the phase space where the Θ function does scale perfectly. We did so by demanding $\mathcal{K}_4=0$ (or $|\vec{\Delta}|=1$) in $\mathcal{S}_4^f$, which can be implemented by simply inserting $\delta(\mathcal{K}_4)$. Here, the analog of that additional constraint is $\mathcal{K}_2=0$. This new condition can also be written as

$$|\vec{K}+\vec{q}| = K_F. \tag{56}$$

Thus, besides staying within their respective cutoffs, the choices available to $\vec{K}$ and $\vec{q}$, when $\mathcal{K}_2=0$, are restricted in such a way that their sum vector must sit precisely on the Fermi surface. Once $\vec{K}$ is chosen, $\vec{q}$ is obligated to connect $\vec{K}+\vec{q}$ to the Fermi surface thus limiting its permissible magnitudes and angles quite severely. This is depicted in Fig. 2.

Under the restriction $\mathcal{K}_2=0$, the boson-fermion coupling can be written as

$$\mathcal{S}_3^{bf} = s^{z+2-d/2z} g \int^{\Lambda} d^d q' \, dk' \, d^{d-1}\Omega_{\vec{K}} d\epsilon' \, d\omega' \, \bar{\psi}' \, \psi' \, \phi' \times \Theta(\Lambda/s - |\mathcal{K}_2|)\, \delta(\mathcal{K}_2) \tag{57}$$

$$= s^{z+2-d/2z} g \int^{\Lambda} d^d q' \, dk' \, d^{d-1}\Omega_{\vec{K}} d\epsilon' \, d\omega' \, \bar{\psi}' \, \psi' \, \phi' \times \delta(k' + s^{(z-1)/z} q' \cos\theta_{Kq}). \tag{58}$$

This should be compared with Eq. (39). In the pure-fermion case, we showed there that whether or not we insert $\delta(\mathcal{K}_4)$ makes no difference to the value of $[u_f]$ because $\delta(\mathcal{K}_4)$ is $\sim\delta(\theta_3-\theta_1)$ or $\sim\delta(\theta_3-\theta_2)$, which has zero scaling dimension. Furthermore, this additional constraint is of a nonsingular nature.

In contrast, for the boson-fermion coupling in Eq. (58), the insertion of $\delta(\mathcal{K}_2)$ involves a dimensionful quantity. If we were to integrate against $\delta(\mathcal{K}_2)$ and eliminate θ_q as suggested by Fig. 2, we would induce an additional 1/momentum factor in violation of the RG edict that the coupling be a nonsingular function of momentum. More intuitively, Fig. 2 shows that imposing the constraint $\mathcal{K}_2=0$ singles out an unrealistic sort of coupling that glues the outgoing fermion $\bar{\psi}_{\vec{K}+\vec{q}}$ to the Fermi surface regardless of the value of $\vec{K}$ or $\vec{q}$. This no longer represents a generic forward scattering process and is of no interest to us. How to correctly capture a generic forwarding scattering process will be discussed in Sec. VI.

At this point, a few issues are worth emphasizing.

(1) Since we integrated against the delta functions in Eq. (41), energy and momentum are explicitly conserved.

(2) The quantity $\mathcal{K}_2$ is not a free variable and it does not necessarily scale in the same way as bosonic or fermionic momenta. This is consistent with the nonscaling of $\mathcal{K}_4$ in the pure-fermion problem when $|\vec{\Delta}|\neq 1$. Only when $z=1$ does $\mathcal{K}_2$, and thus the constraint $\Theta(\Lambda-|\mathcal{K}_2|)$, scale in a simple way.

(3) In this scheme, all components of $\vec{q}$ scale the same way. In particular, $[d^d q]=d/z=d$. At the same time, only fermionic momenta in the direction normal to the Fermi surface scale.

(4) Here, k is not a vector. It does not have parallel or perpendicular components as discussed in certain patching schemes. For more on the patching scheme, see Sec. VI.

(5) Although the RG scheme developed in this section does not work for general values of z, it is perfectly well suited to the special case $z=1$.

(6) Figure 2 gives us an important hint about what may be happening for $z>1$. Since $k'=sk$ and $q'=s^{1/z}q$, we know that after several iterations of the RG, the deviation of $\vec{K}$ from the Fermi surface will be much smaller than the magnitude of $\vec{q}$, i.e., $k\ll q$. As a result $\vec{q}$ will tend to point in a direction perpendicular to $\vec{K}$, which means it will be very nearly tangent to the Fermi surface. In this way, it may seem as if bosonic momenta scale anisotropically in a local coordinate system defined with respect to the direction determined by a fixed $\vec{K}$. This important observation will be developed more fully in Sec. VI when we devise a scheme suitable to $z\neq 1$.

V. CHOICE OF MOMENTUM INTEGRATION

Before moving on to the general case $z\neq 1$, in this section we resolve a seeming ambiguity for the scheme we developed in the previous section. As we found in Eqs. (42) and

(43) there are two ways to express $\mathcal{S}_3^{bf}$ in momentum space. We have already shown in detail that making the choice in Eq. (42) can yield a consistent RG prescription. Now we will show why the alternative decomposition

$$\int^{\Lambda} d^dK_1 d^dK_2[\bar{\psi}_{\vec{K}_2}\psi_{\vec{K}_1}\phi_{K_2-K_1}g(\vec{K}_2,\vec{K}_1)\Theta(\Lambda-|\vec{\mathcal{Q}}|)]$$

is not an appropriate starting point to determine the scaling dimension of the boson-fermion coupling. The problem is that the argument of the boson field, $\vec{\mathcal{Q}}\equiv\vec{K}_1-\vec{K}_2$, does not transform homogeneously, so we do not know what dimension to assign to the boson itself. To see this, write each fermion momentum vector in terms of a direction and a deviation from the Fermi surface: $\vec{K}_i=(K_F+k_i)\hat{K}_i$. This gives

$$\begin{aligned}|\vec{\mathcal{Q}}| &= [K_1^2+K_2^2-2K_1K_2\cos\theta_{12}]^{1/2}\\ &= [(K_F+k_1)^2+(K_F+k_2)^2\\ &\quad -2(K_F+k_1)(K_F+k_2)\cos\theta_{12}]^{1/2}\\ &\approx K_F\sqrt{2}\left[(1-\cos\theta_{12})\left(1+\frac{k_1+k_2}{K_F}\right)\right]^{1/2}\\ &\approx K_F\sqrt{2(1-\cos\theta_{12})}\left(1+\frac{k_1+k_2}{2K_F}\right),\end{aligned} \tag{59}$$

which is true to leading order in $1/N_\Lambda$, and where

$$\cos\theta_{12}=\hat{K}_1\cdot\hat{K}_2 \equiv \begin{cases}\cos(\theta_1-\theta_2)\\ \cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_1\cos(\varphi_1-\varphi_2),\end{cases} \tag{60}$$

for $d=2$ and $d=3$. In Eq. (23) we committed to a specific prescription in making $\mathcal{S}_2^f$ scale invariant where angular components of the momentum do not scale. We therefore cannot allow angles to scale in $\mathcal{S}_3^{bf}$ either. Using the specific prescription in Eq. (46) determined by the quadratic parts of the action we find

$$\begin{aligned}\mathcal{S}_3^{bf} = \int^{\Lambda} & s^{-1}dk_1's^{-1}dk_2's^{-1}d\epsilon_1's^{-1}d\epsilon_2'\Big[s^{3/2}\bar{\psi}'(K_F\\ &+k_2',i\epsilon_2')s^{3/2}\psi'(K_F+k_1',i\epsilon_1')\\ &\times\phi\Big(K_F\sqrt{2(1-\cos\theta_{12})}\left[1+\frac{k_1'+k_2'}{2sK_F}\right],\\ &s^{-1}i\epsilon_2'-is^{-1}\epsilon_1'\Big)g(2,1)\Theta(\Lambda/s^{1/z}-|\vec{\mathcal{Q}}|)\Big],\end{aligned} \tag{61}$$

where

$$\Theta(\Lambda/s^{1/z}-|\vec{\mathcal{Q}}|)\equiv\Theta\left(\Lambda-s^{1/z}K_F\sqrt{2(1-\cos\theta_{12})}\times\left[1+\frac{k_1'+k_2'}{2sK_F}\right]\right). \tag{62}$$

Notice that in Eq. (61) the fermion fields are primed whereas the boson field is not.

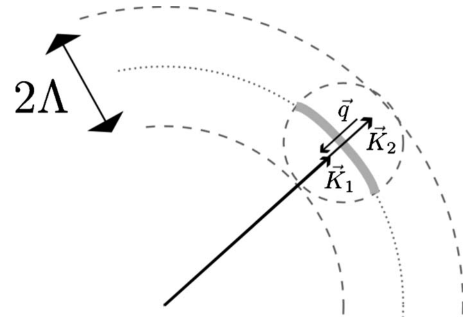

FIG. 3. Under the restriction $\hat{K}_1=\hat{K}_2$, the boson momentum $\vec{q}$ can still have a nonzero modulus, but it loses its angular freedom and is forced to point exactly normal to the Fermi surface. This constraint does not allow the boson field ϕ to transform in a homogeneous fashion.

There are two problems. First, the Θ function does not return to its original form, making it impossible to compare the flow of the coupling function before and after the RG transformation. This is the same problem we encountered in the pure-fermion case of Sec. III and the other boson-fermion prescription from Sec. IV.

Second, we have a new dilemma, which is that we do not know how the ϕ field transforms under the change in argument in Eq. (61). All we know from Eq. (11) is that

$$\phi'(q',i\omega')\equiv s^{-(d+z+2)/(2z)}\phi(s^{1/z}q,si\omega), \tag{63}$$

which states that the boson scales in a (generalized) homogeneous fashion. If we transform the boson arguments in a nonhomogeneous way, as in Eq. (61), we are not guaranteed that such a coordinate transformation will induce a simple multiplicative prefactor. Note that the mathematical requirement that the boson field transform homogeneously means that the relative angle between the incoming and outgoing fermions must be allowed to scale. Said another way, when the magnitude of $\vec{q}$ scales, the angle of $\vec{K}_2=(\vec{K}_1+\vec{q})$ must change. However, when we choose to work in representation, Eq. (43), all momenta are fermionic which forces the wrong type of rescaling on the boson field.

Thus, we cannot adopt the representation in Eq. (43), because, first, the Θ function is not form invariant, and second, it forces a nonhomogeneous coordinate transformation on the boson field. How might we try to remedy these two problems? We could attempt the same strategy that worked in the pure-fermion case where we restricted our consideration to $\mathcal{K}_4=0$; see Eqs. (33)–(35). However, when both bosons and fermions are present this tactic is bound to fail. We already found this in Sec. IV, where we considered the limit $\mathcal{K}_2=0$ using the representation in Eq. (42). Here, the analogous restriction is $\vec{\mathcal{Q}}=0$. Under these circumstances, equivalent to $\vec{K}_1=\vec{K}_2$, the Θ function is trivially invariant. However, the boson loses its field character with $\phi(0)$ not scaling at all.

Let us attempt a different remedy by relaxing the restriction slightly and consider $\hat{K}_1=\hat{K}_2$. This is equivalent to $\mathcal{Q}=|k_1-k_2|$. Here, the Θ function will not be form invariant because it transforms to

$$\Theta(\Lambda - s^{1/z}|k_1 - k_2|) = \Theta(\Lambda - s^{(1-z)/z}|k_1' - k_2'|).$$

Undaunted, we make the further restriction to $z=1$, in which case the Θ function is form invariant, and now less trivially so. Unfortunately, we still have the problem that the boson field scales unnaturally: $\phi(|\vec{\mathcal{Q}}|,\theta_q,\varphi_q)=\phi(|k_1-k_2|,\text{const},\text{const})$. As show in Fig. 3, the boson momentum vector, which is defined as the vector joining the tips of $\vec{K}_1$ and $\vec{K}_2$, lies directly parallel to $\hat{K}_1=\hat{K}_2$. When we force $\hat{K}_1=\hat{K}_2$, the boson momentum loses its angular freedom and thus no longer scales homogeneously.

One might wonder why we are being so strict about the form of the field transformation when it seems like the other scheme in Eq. (42),

$$\int^{\Lambda} d^dKd^dq[\bar{\psi}_{\vec{K}+\vec{q}}\psi_{\vec{K}}\phi_{\vec{q}}g(\vec{K},\vec{q})\Theta(\Lambda-|\mathcal{K}_2|)] \tag{64}$$

also violates this principle. In fact, the fermion is not required to be a homogeneous function of momentum anyway. All that we need from Eq. (23) is

$$\psi(K_F + s^{-1}k') = s^{3/2}\psi'(K_F + k'). \tag{65}$$

The incoming fermion in Eq. (64) is clearly of this form, whereas the outgoing fermion can be written as

$$\bar{\psi}(|\vec{K}+\vec{q}|) \approx \bar{\psi}(K_F + s^{-1}k' + s^{-1}q' \cos\theta_{Kq}). \tag{66}$$

In this form, we know this expression is equivalent to

$$\bar{\psi}(K_F + s^{-1}k' + s^{-1}q' \cos\theta_{Kq}) = s^{3/2}\bar{\psi}'(K_F + k' + q' \cos\theta_{Kq}). \tag{67}$$

Thus, both fermions in Eq. (64) transform as Eq. (23). Finally, $\phi(\vec{q},i\omega)$ obviously scales according to Eq. (63) [and Eq. (11)]. Therefore, in Eq. (64) we know how all fields transform under Eq. (46), so the representation in Eq. (42), suffers from none of the shortcomings we identified for Eq. (43).

With this new understanding, we should also check that in the pure-fermion problem the field $\psi(\mathcal{K}_4)$ transforms in a consistent manner. To see this, we need to keep a few more higher-order terms than what we showed earlier,

$$\begin{aligned}|\vec{K}_4|^2 &= K_1^2 + K_2^2 + K_3^2 + 2\vec{K}_1\cdot\vec{K}_2 - 2\vec{K}_1\cdot\vec{K}_3 - 2\vec{K}_2\cdot\vec{K}_3\\
&\approx 2K_F(k_1+k_2+k_3+3K_F/2) + 2K_F\hat{K}_1\cdot\hat{K}_2(k_1+k_2+K_F)\\
&\quad - 2K_F\hat{K}_1\cdot\hat{K}_3(k_1+k_3+K_F) - 2K_F\hat{K}_2\cdot\hat{K}_3(k_2+k_3+K_F)\\
&= 2K_F\{K_F(\hat{K}_1\cdot\hat{K}_2 - \hat{K}_1\cdot\hat{K}_3 - \hat{K}_2\cdot\hat{K}_3 + 3/2)\\
&\quad + k_1[1+\hat{K}_1\cdot(\hat{K}_2-\hat{K}_3)] + k_2[1+\hat{K}_2\cdot(\hat{K}_1-\hat{K}_3)]\\
&\quad + k_3[1-\hat{K}_3\cdot(\hat{K}_1+\hat{K}_2)]\}\\
&= 2K_F\{K_F|\vec{\Delta}|^2/2\\
&\quad + k_1[1+\hat{K}_1\cdot(\hat{K}_2-\hat{K}_3)] + k_2[1+\hat{K}_2\cdot(\hat{K}_1-\hat{K}_3)]\\
&\quad + k_3[1-\hat{K}_3\cdot(\hat{K}_1+\hat{K}_2)]\},\end{aligned} \tag{68}$$

where we used Eq. (31). In the special case where $\hat{K}_1=\hat{K}_3$, corresponding to forward scattering, we have

$$|\vec{K}_4| \approx K_F + k_2 + (k_1 - k_3)\hat{K}_1\cdot\hat{K}_2. \tag{69}$$

This shows that

$$\psi(|\vec{K}_4|) = \psi(K_F + s^{-1}k_2' + s^{-1}(k_1'-k_3')\hat{K}_1\cdot\hat{K}_2), \tag{70}$$

which is precisely the scaling form appropriate for a fermion in Eq. (23). In the same way, it is easy to show that the fermion scales appropriately for the cases $\hat{K}_2=\hat{K}_3$ and $\hat{K}_1=-\hat{K}_2$. In these cases we have

$$|\vec{K}_4| \approx K_F + k_1 + (k_2 - k_3)\hat{K}_1\cdot\hat{K}_3, \tag{71}$$

$$|\vec{K}_4| \approx K_F + k_3 + (k_2 - k_1)\hat{K}_1\cdot\hat{K}_3, \tag{72}$$

respectively. Thus, all the results of Shankar remain valid.

Finally, it may at first seem puzzling that the scaling of the constraint in Eq. (62) is so problematic since we were able to find a simple solution in the pure-fermion problem involving $\mathcal{S}_4^f$. There, the constraint involved $\mathcal{K}_4=|\vec{K}_3-\vec{K}_2-\vec{K}_1|-K_F$, which measures a deviation from the Fermi surface. However, in the representation of the boson-fermion coupling in Eq. (43), the constraint involves $\vec{\mathcal{Q}}=\vec{K}_2-\vec{K}_1$ which is not a deviation from the Fermi surface and as written, can take any value between 0 and $2K_F$; see Eq. (59).

To summarize, $\psi(\mathcal{K}_4)$ scales like a fermion, $\psi(\mathcal{K}_2)$ scales like a fermion, but $\phi(\vec{\mathcal{Q}})$ does not scale like a boson. We therefore cannot use Eq. (43) to represent the boson-fermion coupling because we do not have knowledge of the boson field scaling under such a coordinate transformation.

VI. PATCHING SCHEME

When $z\neq 1$, the scheme we developed in Sec. IV no longer works. The problem is that under mode elimination and rescaling, the constraint function changes its form,

$$\begin{aligned}\Theta(\Lambda/s - |\mathcal{K}_2|) &\approx \Theta(\Lambda/s - |k + q\cos\theta_{Kq}|)\\
&= \Theta(\Lambda - s|k+q\cos\theta_{Kq}|)\\
&= \Theta(\Lambda - |k' + s^{(z-1)/z}q'\cos\theta_{Kq}|),\end{aligned} \tag{73}$$

where $k'=sk$ and $q'=s^{1/z}q$. When $z\neq 1$, we cannot reliably determine the flow of the coupling because the structure of the interaction itself has changed under this RG transformation. This is the same dilemma encountered in the pure-fermion problem in Eq. (32). Also, notice that writing $\mathcal{S}_3^{bf}$ in terms of an integral over $\vec{K}_1$ and $\vec{K}_2$, rather than $\vec{K}$ and $\vec{q}$, will not cure the problem. In the previous section we explained why this is the case, even for $z=1$.

For these reasons, when $z\neq 1$ we adopt a different method where we scale toward a specific point on the Fermi surface. Although the details differ, this is similar in spirit to some previous work on the renormalization of the gauge-spinon problem.[9–12]

In $d=2$, consider the annulus in momentum space defined by $-\Lambda<k<\Lambda$. Now subdivide the annulus into N_Λ regions of angular size $\Delta\theta$: $N_\Lambda \Delta\theta = 2\pi \Rightarrow \Delta\theta = 2\pi\Lambda/K_F$. Each patch will be approximately of size $\sim\Lambda^2$. The same idea is easily generalized to $d>2$. This should be familiar from multidimensional bosonization[27] and functional RG;[16–19] we refer the reader to those papers for further details.

The momentum integral for the quadratic part of the fermionic action, $\mathcal{S}_2^f$, can now be decomposed into a sum over N_Λ identical patches,

$$\mathcal{S}_2^f = \int^\Lambda d^dK d\epsilon \bar{\psi}(i\epsilon - \xi_{\vec{K}})\psi = \sum_{p=1}^{N_\Lambda} \int^\Lambda d^dk_p d\epsilon_p \bar{\psi}_p(i\epsilon_p - \xi_{\vec{k}_p})\psi_p. \tag{74}$$

Here, $\vec{k}_p = (\vec{k}_{p,\perp}, k_{p,\parallel})$ is a local coordinate within each patch which has components parallel and perpendicular to some reference frame. We define this special local reference direction to be the normal vector to the Fermi surface at the patch origin. Thus, $\vec{k}_\perp$ is tangent to the Fermi surface at the patch origin. Within each patch, functions of momentum can be expanded around the patch origin. Consider, for example, the patch centered at $\vec{K} = (0, K_F)$ which we will label as patch $p=1$, and where we have specialized to $d=2$ for concreteness. Near this point, the dispersion of a perfectly parabolic band can be expressed in terms of local patch coordinates as follows:

$$\begin{aligned}\xi_{\vec{K}\approx(0,K_F)} &\approx \frac{1}{2m}\bigg[(K_x^2+K_y^2)|_{(0,K_F)} + 2K_x|_{(0,K_F)}(K_x-0) \\ &\quad + 2K_y|_{(0,K_F)}(K_y-K_F) + \frac{1}{2m}\frac{1}{2}2|_{(0,K_F)}(K_x-0)^2 \\ &\quad + \frac{1}{2m}\frac{1}{2}2|_{(0,K_F)}(K_y-K_F)^2\bigg] - \frac{K_F^2}{2m} \\ &\approx v_F k_{1,\parallel} + \frac{k_{1,\perp}^2}{2m} \\ &= v_F k_{1,\parallel} + \frac{v_F k_{1,\perp}^2}{2K_F} \\ &\equiv v_F k_{1,\parallel} + a k_{1,\perp}^2,\end{aligned} \tag{75}$$

where for this particular patch, $k_{1,\parallel} \equiv K_y - K_F$, and $k_{1,\perp} \equiv K_x$. We have also defined $v_F \equiv K_F/m$ and $a \equiv v_F/(2K_F) = 1/(2m)$. As a sum over all the patches that enclose the Fermi surface, the quadratic part of the action can now be written as

$$\mathcal{S}_2^f = \sum_{p=1}^{N_\Lambda} \int^\Lambda d^dk_p d\epsilon_p \bar{\psi}_p(i\epsilon_p - v_F k_{p,\parallel} - a\vec{k}_{p,\perp}^2)\psi_p.$$

Note that the concepts of parallel and perpendicular only make sense with respect to a perfectly flat surface, or the normal to a specific point on a curved surface. We take this specific point to be the center of the patch. Momentum components in the same direction as the vector normal to the Fermi surface at the patch origin are considered "parallel," whereas momenta tangent to the Fermi surface are labeled "perpendicular." We caution that different conventions exist in the literature for what is deemed parallel or perpendicular. We adopt the convention of Ref. 11.

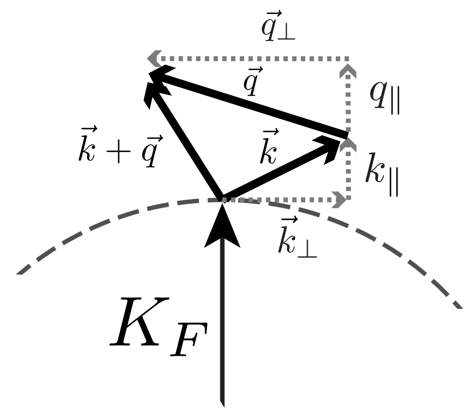

FIG. 4. Local coordinate system in patch $p=1$ whose center is located at $\vec{K} = (0, K_F)$. Bosonic and fermionic momenta now scale identically.

Within each patch, the momentum integral is limited to a box of dimension Λ in every direction. For example, in $d=2$ this means

$$\int^\Lambda d^2k \equiv \int_{-\Lambda}^{\Lambda} dk_\parallel \int_{-\Lambda}^{\Lambda} dk_\perp. \tag{76}$$

We have dropped the patch indices since we assume all patches are identical in the sense that variables scale in the same manner in every patch. In the absence of van Hove singularities and nesting instabilities, this is a reasonable assumption.[28]

Within this patching formalism, and when we consider only $\vec{q} \approx 0$ so that the entire boson phase space can be restricted to a single patch, the quadratic part of the bosonic action can be written in straightforward fashion. To be concrete, consider $z=3$,

$$\mathcal{S}_2^b = \int^\Lambda d^dq d\omega \phi^* \left(\vec{q}_\perp^2 + q_\parallel^2 + \frac{\gamma\omega}{\sqrt{\vec{q}_\perp^2 + q_\parallel^2}}\right)\phi.$$

Within each patch, bosonic momenta $\vec{q}$ and fermionic momenta $\vec{k}$ are all measured with respect to the same single point, the patch origin. Consequently, bosonic and fermionic momenta scale the same way, that is

$$[\vec{k}_\perp] = [\vec{q}_\perp], \tag{77}$$

$$[k_\parallel] = [q_\parallel]. \tag{78}$$

See Fig. 4. Whether we label momenta by $\vec{k}$ or $\vec{q}$ is thus immaterial since they scale identically; this is in stark contrast to the scheme developed in Sec. IV for the case $z=1$.

Of course, the possibility exists that $[k_\parallel] \neq [\vec{k}_\perp]$. In fact, we will now argue why they *cannot* be the same when $z \neq 1$.

The fixed point is defined by constructing the scaling scheme so that the quadratic part of the action, $\mathcal{S}_2^b + \mathcal{S}_2^f$, is scale invariant. Scale invariance of $\mathcal{S}_2^f$ requires

$$[\epsilon]=[v_F]+[k_\parallel]=[a]+2[\vec{k}_\perp] \tag{79}$$

while scale invariance of $\mathcal{S}_2^b$ necessitates

$$[\omega]+[\gamma]=[(\vec{q}_\perp^{\,2}+q_\parallel^2)^{z/2}]. \tag{80}$$

Next, we observe that since we want to scale bosons and fermions simultaneously, it is sensible to give their energies equal scaling dimensions which we denote by

$$[E]\equiv[\omega]=[\epsilon]. \tag{81}$$

The conditions from the quadratic parts of the action now become

$$[E]=[v_F]+[k_\parallel]=[a]+2[\vec{k}_\perp], \tag{82}$$

$$[E]+[\gamma]=[(\vec{q}_\perp^{\,2}+q_\parallel^2)^{z/2}]. \tag{83}$$

At this point we demand that the dispersion relations of low-energy excitations be preserved under scaling. For fermions near the Fermi surface, energy must be a linear function of momentum. We thus set $[v_F]=0$ to establish

$$[E]=[k_\parallel]=[a]+2[\vec{k}_\perp]. \tag{84}$$

This is furthermore justified by the fact that $\frac{ak_\perp^2}{v_F k_\parallel}=\frac{k_\perp^2}{K_F k_\parallel}\ll 1$ provided $k_{\perp,\parallel}\leq\Lambda\ll K_F$. The "curvature" term is a small correction. Thus, parallel momenta scale like energy.

We now use Eqs. (78) and (84) in Eq. (83) to determine the dimension of the perpendicular momentum,

$$[\gamma E]=[(\vec{q}_\perp^{\,2}+v_F^2E^2)^{z/2}]\Rightarrow[\vec{q}_\perp]=[\sqrt{(\gamma E)^{2/z}-v_F^2E^2}]. \tag{85}$$

In the infrared limit, this becomes

$$[\vec{q}_\perp]=([E]+[\gamma])/z \tag{86}$$

because $E^{2/z}>E^2$ when $z>1$. To preserve the bosonic dispersion (i.e., $\omega\sim q^z$), we set $[\gamma]=0$, obtaining

$$[\vec{q}_\perp]=[E]/z. \tag{87}$$

Now we plug this result into Eq. (84) to find the dimension of a,

$$[E]=[a]+2[E]/z\Rightarrow[a]=(1-2/z)[E]. \tag{88}$$

Finally, we are free to choose the value of $[E]$, which we set equal to unity for convenience; any other value will only induce the same multiplicative prefactor on all dimensions, but relative dimensions will be unaffected. To summarize,

$$[E]=[\epsilon]=[\omega]=1,$$

$$[k_\parallel]=[q_\parallel]=[E]=1,$$

$$[\vec{k}_\perp]=[\vec{q}_\perp]=[E]/z=1/z,$$

$$[a]=1-2/z,$$

$$[v_F]=0,$$

$$[\gamma]=0. \tag{89}$$

Note that for $z=3$ the dimension of $[\vec{k}_\perp]$ appears to suggest that the fermionic band structure changes under scaling. This is an illusion since the parameter a, which is a measure of the curvature, is allowed to scale in order to precisely compensate the scaling of $\vec{k}_\perp$, thus ensuring that the band remains invariant.

Plugging these values into the quadratic action yields the dimensions of the fields,

$$[\psi]=-\frac{3z+d-1}{2z}, \tag{90}$$

$$[\phi]=-\frac{2z+d+1}{2z}. \tag{91}$$

We now have enough information to determine the dimension of the boson-fermion coupling,

$$\begin{aligned}\mathcal{S}_3^{bf}=&\sum_{\text{patches}}\int d^dk_1d^dk_2d^dqd\epsilon_2d\epsilon_1d\omega\\&\times g(\vec{k}_2,\vec{k}_1,\vec{q},\epsilon_2,\epsilon_1,\omega)\bar{\psi}(2)\psi(1)\phi(\vec{q},\omega)\\&\times\delta^{(d-1)}(\vec{k}_{2,\perp}-\vec{k}_{1,\perp}-\vec{q}_\perp)\delta(k_{2,\parallel}-k_{1,\parallel}-q_\parallel)\delta(\epsilon_2-\epsilon_1-\omega)\\&\times\Theta(\Lambda-|\vec{k}_{2,\perp}|)\Theta(\Lambda-|k_{2,\parallel}|)\Theta(\Lambda-|\vec{k}_{1,\perp}|)\Theta(\Lambda\\&-|k_{1,\parallel}|)\Theta(\Lambda-|\vec{q}_\perp|)\Theta(\Lambda-|q_\parallel|).\end{aligned} \tag{92}$$

Note that this is slightly less general than could be the case. We have restricted our consideration to nearly forward scattering processes which means that $\vec{q}\approx 0$ or, equivalently, $\vec{k}_1$ and $\vec{k}_2$ belong to the same patch. Interpatch processes, such as the BCS instability, are not captured.

Since we are scaling toward a single point, momentum and energy conserving delta functions and cutoff constraints factorize nicely. Integrating against the delta functions yields

$$\begin{aligned}\mathcal{S}_3^{bf}=&\sum_{\text{patches}}\int^\Lambda d^dkd^dqd\epsilon d\omega g(\vec{k},\vec{q},\epsilon,\omega)\bar{\psi}(\vec{k}+\vec{q},\epsilon+\omega)\\&\times\psi(\vec{k},\epsilon)\phi(\vec{q},\omega)\Theta(\Lambda-|\vec{k}_\perp+\vec{q}_\perp|)\Theta(\Lambda-|k_\parallel+q_\parallel|),\end{aligned} \tag{93}$$

where we have placed some of the constraints in the limits of integration. Unlike what happened in Sec. IV, there is no difference in eliminating boson or fermionic variables due to Eqs. (89). Here it is arbitrary whether we call momentum $\vec{k}$ or $\vec{q}$ since in the patching scheme they scale the same way. Additionally, the factorization of parallel and perpendicular components of momenta means the arguments of the fields scale in a straightforward fashion. Indeed, after mode elimination and rescaling we find,

$$\mathcal{S}_3^{bf} = g \sum_{\text{patches}} \int^{\Lambda} s^{-1} s^{-(d-1)/z} d^d k' s^{-1} s^{-(d-1)/z} d^d q'$$
$$\times s^{-1} d\epsilon' s^{-1} d\omega' s^{(3z+d-1)/(2z)} \bar{\psi}' s^{(3z+d-1)/(2z)} \psi' s^{(2z+d+1)/(2z)}$$
$$\times \phi' \Theta(\Lambda/s^{1/z} - s^{-1/z}|\vec{k}'_\perp + \vec{q}'_\perp|)\Theta(\Lambda/s - s^{-1}|k'_\parallel + q'_\parallel|)$$
$$= s^{(3-d)/(2z)} g \sum_{\text{patches}} \int^{\Lambda} d^d k' d^d q' d\epsilon' d\omega' \bar{\psi}' \psi' \phi'$$
$$\times \Theta(\Lambda - |\vec{k}'_\perp + \vec{q}'_\perp|)\Theta(\Lambda - |k'_\parallel + q'_\parallel|), \tag{94}$$

where we have Taylor expanded g and kept the most relevant (constant) piece. In this patching scheme, the constraints and fields transform in a simple way, so we can simply read off the dimension of the coupling,

$$[g] = \frac{3-d}{2z}. \tag{95}$$

The relevance or irrelevance of this coupling is in some sense arbitrary outside the context of a specific physical problem. The value of $[g]$ depends crucially on the dimensions $[\phi]$ and $[\psi]$, and these will be determined by the problem under consideration. For example, in the context of magnetic phases of the Kondo lattice, see Refs. 22–24.

Several important comments are now in order:

(1) the result in Eq. (95) is identical to Eq. (55) when $z=1$. Therefore, the patching scheme developed in this section yields an answer equivalent to the extension of Shankar's scheme presented in Sec. IV using global coordinates. While the latter approach is perhaps more intuitive, it is not justifiable when $z \neq 1$. On the other hand, the patching scheme requires same careful interpretation, as discussed below, but is consistent for any value of z.

(2) It is necessary to give the curvature parameter, a, a nonzero scaling dimension in order to compensate for the way that $k_\perp$ scales. Rest assured, however, that $[ak_\perp^2]=[v_F k_\parallel]=[\epsilon]$ so that the fermion band is kept invariant. In this way, we do not need to scale the number of patches.

(3) It may seem as if the bosons have become anisotropic but this is an illusion due to the nature of the local coordinates we have chosen. Because of the sum over patches, we have included an equal weighting of $\vec{q}$ components in all directions, even though locally we only keep $\vec{q}_\perp$ within each patch. Of course, it does mean that in the low-energy limit bosons become locally tangent to the Fermi surface for fixed value of fermionic momentum $\vec{K}$. This is not surprising and was noticed long ago.[9,11] We even saw hints of this in Sec. IV. In that scaling scheme $k'=sk$ and $q'=s^{1/z}q$. When $z>1$, the length of $|\vec{q}|$ scales more slowly than the deviation from the Fermi surface, k. As a result, in the low-energy limit, the boson momentum will tend to lie tangent to the Fermi surface.

(4) In the patching formalism, the dimension of the boson field in Eq. (91) derives from

$$\phi'(q'_\parallel, \vec{q}'_\perp, i\omega') = s^{[\phi]}\phi(s^{-[q_\parallel]}q'_\parallel, s^{-[\vec{q}_\perp]}\vec{q}'_\perp, s^{-[\omega]}i\omega')$$

and similarly for the fermion field. Once again this takes the form of a generalized homogeneous function but is different from the type of scaling in Eq. (11) and (23). For more on generalized homogeneous functions, see Ref. 29.

(5) The form of the interaction we consider is limited to nearly forward scattering ($q \approx 0$) intrapatch processes. Interpatch processes are not captured and this makes comparisons with the pure-fermion RG somewhat delicate. Consider a four-fermion interaction with incoming momenta $\vec{K}_1$ and $\vec{K}_2$, and outgoing momenta $\vec{K}_3$ and $\vec{K}_4$. The difference between incoming and outgoing momenta at the left vertex can be small, say $\vec{K}_3 - \vec{K}_1 \equiv \vec{q}_{\text{left vertex}} \approx 0$. This can match up with small momentum transfer on the right: $\vec{K}_4 - \vec{K}_2 \equiv \vec{q}_{\text{right vertex}} \approx 0$. However, this says nothing about the relationship between $\vec{K}_1$ and $\vec{K}_2$. Indeed, $\vec{K}_1$ and $\vec{K}_2$ can each independently take any value around the Fermi surface, i.e., $|\vec{K}_2 - \vec{K}_1|$ can take any value between 0 and $2K_F$. Thus, "forward scattering" processes in a boson-fermion formalism are not necessarily equivalent to forward scattering processes in a four-fermion formalism. The latter (four-fermion coupling) involves two patches, whereas the former (boson-fermion coupling) involves only one patch. In other words, the dimension of u_f is not simply given by $[g^2]$.

(6) If we were to include self-energy corrections into $\mathcal{S}_2^f$ and establish this as the new fixed point, the values of the dimension assignments would change, but the philosophy would be the same. For example, in the gauge-spinon[11] and ferromagnetic Kondo lattice systems,[24] gapless overdamped $z=3$ bosons lead to a characteristic electron self-energy $\Sigma(\epsilon) \sim \epsilon^{2/3}$ in $d=2$ and $\Sigma(\epsilon) \sim -\epsilon \log \epsilon$ in $d=3$. We can define the new fixed-point action with $\mathcal{S}_2^f = \int \bar{\psi}(\epsilon^{d/z} - v_F k_\parallel - ak_\perp^2)\psi$. Using the same philosophy defined in this section, we would assign

$$[E] = [\epsilon] = [\omega] = z/d,$$
$$[k_\parallel] = [q_\parallel] = [E] = 1,$$
$$[\vec{k}_\perp] = [\vec{q}_\perp] = [E]/z = 1/d,$$
$$[a] = 1 - 2/d,$$
$$[v_F] = 0,$$
$$[\gamma] = 0. \tag{96}$$

This also leads to a change in the dimensions of the fields and the couplings but the methodology is no different than what has already been discussed above. See Ref. 24 for further discussion.

(7) There is some debate in the literature about how to properly scale the gauge-spinon model which corresponds to $z=3$.[9–12,30] The consistent scaling scheme within the patching formalism we advocate here coincides with that of Ref. 11.

VII. CONCLUSION

This paper has developed an easy-to-use RG procedure for theories containing both bosons and fermions with a Fermi surface. We reviewed the global coordinate approach to the fermionic RG as formulated by Shankar, showed how to generalize this formalism to include bosons with dynamical exponent $z=1$, and explained why such an approach will not work when $z\neq 1$. We pointed out that a consistent scheme must ensure that the kinematic constraints, which result from the conservation of momentum and the effective-field theory cutoffs, remain invariant to the RG transformation. In addition, field rescaling can only be properly identified in interaction terms when the coordinates of the field transform in a known way, as specified by the quadratic part of the action.

We also showed that, for $z=1$, the same results arise within a patching scheme. Here the momentum space near the Fermi surface is partitioned into patches. For $z\neq 1$, the patching scheme represents the only consistent RG approach to mixed fermion-boson systems.

Coupled boson and fermion problems arise in a variety of contexts. We have already mentioned the problems of itinerant magnets which have directly motivated our work here, as well as the subject of gauge fields coupled to fermions. In addition, fermion-boson mixtures of cold atomic gases[31] may provide another interesting setting for this work. We hope the RG program described here will be useful for related problems in other settings as well.

ACKNOWLEDGMENTS

We thank R. Shankar for useful discussions on the fermionic RG. This work has been supported in part by the NSF under Grant No. DMR-0706625, the Robert A. Welch Foundation Grant No. C-1411, and the W. M. Keck Foundation. One of us (Q.S.) gratefully acknowledges the Aspen Center for Physics for hospitality.

[1] K. G. Wilson and J. B. Kogut, Phys. Rep. **12**, 75 (1974).
[2] J. A. Hertz, Phys. Rev. B **14**, 1165 (1976).
[3] J. Feldman and E. Trubowitz, Helv. Phys. Acta. **63**, 156 (1990).
[4] G. Benfatto and G. Gallavotti, Phys. Rev. B **42**, 9967 (1990).
[5] R. Shankar, Physica A **177**, 530 (1991).
[6] R. Shankar, Rev. Mod. Phys. **66**, 129 (1994).
[7] T. Vojta, D. Belitz, R. Narayanan, and T. Kirkpatrick, Z. Phys. B: Condens. Matter **103**, 451 (1997).
[8] A. Abanov and A. V. Chubukov, Phys. Rev. Lett. **93**, 255702 (2004).
[9] J. Polchinski, Nucl. Phys. B **422**, 617 (1994).
[10] C. Nayak and F. Wilczek, Nucl. Phys. B **430**, 534 (1994).
[11] B. L. Altshuler, L. B. Ioffe, and A. J. Millis, Phys. Rev. B **50**, 14048 (1994).
[12] M. Onoda, I. Ichinose, and T. Matsui, Nucl. Phys. B **446**, 353 (1995).
[13] T. Holstein, R. Norton, and P. Pincus, Phys. Rev. B **8**, 2649 (1973).
[14] C. M. Varma, Z. Nussinov, and W. v. Saarloos, Phys. Rep. **361**, 267 (2002).
[15] P. Lee, N. Nagaosa, and X. Wen, Rev. Mod. Phys. **78**, 17 (2006).
[16] N. Furukawa, T. M. Rice, and M. Salmhofer, Phys. Rev. Lett. **81**, 3195 (1998).
[17] D. Zanchi and H. J. Schulz, Europhys. Lett. **44**, 235 (1998).
[18] D. Zanchi and H. J. Schulz, Phys. Rev. B **61**, 13609 (2000).
[19] C. Honerkamp, M. Salmhofer, N. Furukawa, and T. M. Rice, Phys. Rev. B **63**, 035109 (2001).
[20] F. Schütz, L. Bartosch, and P. Kopietz, Phys. Rev. B **72**, 035107 (2005).
[21] S. W. Tsai, A. H. Castro Neto, R. Shankar, and D. K. Campbell, Phys. Rev. B **72**, 054531 (2005).
[22] S. J. Yamamoto and Q. Si, Phys. Rev. Lett. **99**, 016401 (2007).
[23] S. J. Yamamoto and Q. Si, Physica B **403**, 1414 (2008).
[24] S. J. Yamamoto and Q. Si, arXiv:0812.0819 (unpublished).
[25] T. Moriya, *Spin Fluctuations in Itinerant Electron Magnetism* (Springer, New York, 1985).
[26] A. J. Millis, Phys. Rev. B **48**, 7183 (1993).
[27] P. Kopietz, *Bosonization of Interacting Fermions in Arbitrary Dimensions* (Springer-Verlag, Cambridge, 1997).
[28] Actually, van Hove singularities may lead to the dominance of some patches over others. Such an idea has been explored, for example, by Ref. 16, where a two-patch model was employed. This has been generalized to N patches in the functional RG literature.
[29] A. Hankey and H. E. Stanley, Phys. Rev. B **6**, 3515 (1972).
[30] S. S. Lee, Phys. Rev. B **78**, 085129 (2008).
[31] K. Yang, Phys. Rev. B **77**, 085115 (2008).